Classical decay of Coulomb charges

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In this paper we postulate and solve the following problem: Prove in the framework of Newtonian mechanics that three Coulomb charges $(-1,Q,Q)$ for $Q>4$ will leave any initial volume in a finite time and estimate this time. We also discuss possible generalizations of the problem and its relation to stability of ions and molecules.

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The classical dynamics of three charges has been discussed in textbooks such as [1]. A well-known scaling property of the Coulomb interaction reduces the decay problem for charges $(-q_1,q_2,q_3)$ to that for charges $(-1,Q_2,Q_3)$. The scaling of charges results in some scaling of space and time coordinates, which leads to the initial Lagrangian multiplied by a factor and thus to the same dynamics [2].

For appropriate initial conditions and $0<Q<1$ the system of charges $(-1,Q,Q)$ can stay infinitely long within a prescribed volume, which we take as a sphere with radius R. This is because a positive charge $Q<1$ can perform an ordinary Kepler motion around the negative charge, while the second positive charge can do the same, since it ''sees'' a monopole attraction of charge $-1+Q$, and its perturbation of the pair $(-1,Q)$ can be assumed negligible. We will show that this is impossible when $Q>4$ and give a lower bound for the time it takes for the charges to leave any given volume. For point charges there is zero probability for one charge to hit another (the closer one charge gets to another, the less it ''feels'' the presence of other charges and its motion becomes just Keplerian) and we assume all trajectories to be continuous and smooth.

Following [3] we draw all coordinates from O, the center of mass (CM) of the three-particle system, which is at rest. We start by calculating the second time derivative of the moment of inertia of the system: $J=m_1\mathbf{r}_1^2+m_2\mathbf{r}_2^2+m_3\mathbf{r}_3^2$, where $(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ are position vectors of the charges $(-1,Q,Q)$ with masses (m_1,m_2,m_3), respectively. We use the notation $m=\min(m_1,m_2,m_3)$ and $M=\max(m_1,m_2,m_3)$:

$$\ddot{J}=\frac{d}{dt}2(m_1\mathbf{r}_1\cdot\dot{\mathbf{r}}_1+m_2\mathbf{r}_2\cdot\dot{\mathbf{r}}_2+m_3\mathbf{r}_3\cdot\dot{\mathbf{r}}_3)$$
$$=4T+2(\mathbf{r}_1\cdot\mathbf{F}_{23}+\mathbf{r}_2\cdot\mathbf{F}_{13}+\mathbf{r}_3\cdot\mathbf{F}_{12}). \quad (1)$$

Here T stands for the total kinetic energy, $\mathbf{F_{ij}}$ is the force acting from the charges i and j on the third charge k, and (ijk) is a permutation of (123). Rewriting the last equation in terms of the potential energy $V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ we obtain

$$\ddot{J}=4T-2(\mathbf{r}_1\cdot\boldsymbol{\nabla}_1V+\mathbf{r}_2\cdot\boldsymbol{\nabla}_2V+\mathbf{r}_3\cdot\boldsymbol{\nabla}_3V), \quad (2)$$

where the potential energy is

$$V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\frac{Q^2}{|\mathbf{r}_2-\mathbf{r}_3|}-\frac{Q}{|\mathbf{r}_1-\mathbf{r}_2|}-\frac{Q}{|\mathbf{r}_1-\mathbf{r}_3|}. \quad (3)$$

For a smooth homogeneous function $f(x)$ of degree k, $f(\beta\mathbf{r})=\beta^kf(\mathbf{r})$, Euler's theorem [2] gives $(\mathbf{r}\cdot\boldsymbol{\nabla}f)=kf(\mathbf{r})$. Since the potential V is a homogenous function of degree -1 we may apply Euler's theorem to Eq. (2) and get

$$\ddot{J}=2T+2E, \quad (4)$$

where $E=T+V$ is the total energy of the system. If E is positive, then inevitably after some time the system will monotonically expand, since the second derivative is positive and greater than some E. Growth in J is obviously equivalent to expansion in Cartesian coordinates, since $J\leqslant 3MR_{max}^2$, where $R_{max}=\max(r_1,r_2,r_3)$. The decay (migration) time t_d is estimated below.

If the energy is negative then the system stays in configurations with negative $V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$. We introduce the instant plane S [see Fig. 1(a)], which is perpendicular to the line connecting the two positive charges and positioned midway between them, and thus separates the whole space into two parts. If the negative charge stays in this plane all three charges form a triangle with two equal sides $r_{12}=r_{13}$ and a third side r_{23}, for which $r_{23}\leqslant 2r_{12}$, [see Fig. 1(b)]. The potential energy then becomes

$$V=\frac{Q^2}{r_{23}}-\frac{2Q}{r_{12}}\geqslant\frac{Q^2}{r_{23}}-\frac{4Q}{r_{23}}>0, \quad (5)$$

i.e., positive if $Q>4$. Thus in the case $E\leqslant 0$ it is impossible for the negative charge to stay in the plane S.

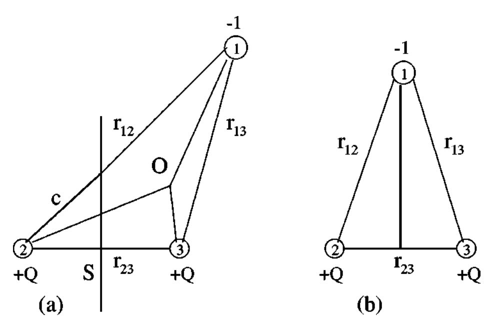

FIG. 1. (a) General positioning of three charges with S being an instant plane midway between positive charges, perpendicular to the line connecting them. (b) The negative charge being positioned in the plane S makes the potential energy positive when $Q>4$ (see text).

This observation drives us to the conclusion that if the negative charge initially was closer to, say, the positive charge labeled 3, it would also subsequently remain closer to it since it cannot penetrate through the plane S. Now we calculate the radial projection $\mathbf{F}_{13}\cdot\hat{\mathbf{r}}_2$ of the force acting on the positive charge labeled 2 (the caret denoting a unit vector in the $\mathbf{r}_2$ direction),

$$\mathbf{F}_{13}\cdot\hat{\mathbf{r}}_2=\frac{Q^2}{r_{23}^2}(\hat{\mathbf{r}}_{23}\cdot\hat{\mathbf{r}}_2)-\frac{Q}{r_{12}^2}(\hat{\mathbf{r}}_{21}\cdot\hat{\mathbf{r}}_2), \tag{6}$$

with

$$\mathbf{r}_{ik}=\mathbf{r}_i-\mathbf{r}_k. \tag{7}$$

Since the center of mass is always located inside the triangle, the two following inequalities obviously hold: $(\hat{\mathbf{r}}_{23}\cdot\hat{\mathbf{r}}_2)\geqslant(\hat{\mathbf{r}}_{23}\cdot\hat{\mathbf{r}}_{21})$ and $(\hat{\mathbf{r}}_{21}\cdot\hat{\mathbf{r}}_2)\leqslant 1$. Thus from Eq. (6) we derive the inequality

$$\mathbf{F}_{13}\cdot\hat{\mathbf{r}}_2\geqslant\frac{Q^2}{r_{23}^2}(\hat{\mathbf{r}}_{23}\cdot\hat{\mathbf{r}}_{21})-\frac{Q}{r_{12}^2}. \tag{8}$$

Charge -1 stays on the right-hand side of the plane S. We can improve the inequality by putting the negative charge on the plane S, keeping the same angle $(\hat{\mathbf{r}}_{23}\cdot\hat{\mathbf{r}}_{21})$, and minimizing r_{12}. Now $(\hat{\mathbf{r}}_{23}\cdot\hat{\mathbf{r}}_{21})=r_{23}/2r_{12}$, and we rewrite the inequality as

$$\mathbf{F}_{13}\cdot\hat{\mathbf{r}}_{12}\geqslant\frac{Q}{r_{23}r_{12}}\left(\frac{Q}{2}-\frac{r_{23}}{r_{12}}\right)\geqslant\frac{Q(Q-4)}{r_{23}^2}>0, \tag{9}$$

where we used $r_{12}\geqslant r_{23}/2$ and $Q>4$. The right-hand side of the first inequality is simply $V/2r_{12}$. From Eq. (9) we see that the force acting on 2 from charges 1 and 3 always has a positive radial projection, which makes charge 2 leave the prescribed volume. The time spent is estimated below.

It is important that $Q>4$ since for $Q=4$ the whole system has an equilibrium position with the negative charge right between the two positive charges. Although unstable, this equilibrium would make untrue our statement about decay regardless of any initial conditions or mass ratio.

To estimate the time expenditure of the decay we have to consider separately the cases $E\geqslant E_0$ and $E<E_0$, where E_0 is some low positive threshold, which is calculated below.

At $E\geqslant E_0$ an infinity of motion is predicted by the well-known virial theorem, which nevertheless does not predict the time when the system leaves the volume (for example, it can be expected greater than the age of the solar system, as in Poincaré's theorem on returning). Here we calculate the time that is the deadline for charges to escape from a given volume, which is independent of any initial conditions except this volume.

For $E\geqslant E_0$ we can write an inequality using Eq. (4):

$$J(t)\geqslant J_0+\dot{J}_0(t-t_0)+E_0(t-t_0)^2, \tag{10}$$

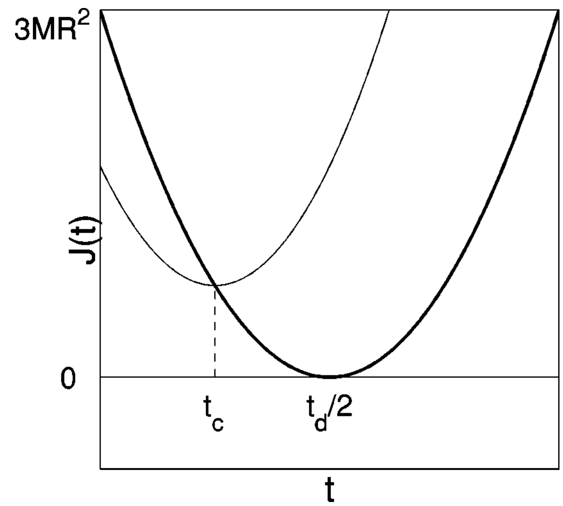

FIG. 2. The sketched actual curve crosses the reference parabola (bold) at some point t_0 only once.

where J_0 and $\dot{J}_0$ are taken from initial conditions at time t_0 and the inequality (10) is valid at $t\geqslant t_0$. Recalling that $J(t_0)\leqslant 3MR^2$ it is now useful to compare the non-negative function $J(t)$ with the parabola (see Fig. 2)

$$f(t)=E_0\left(t-\sqrt{\frac{3}{E_0}MR}\right)^2, \tag{11}$$

which touches the t axis at the point $t_d/2=R\sqrt{3M/E_0}$ and has the curvature $\ddot{f}(t)=2E_0$. Since $f(0)=3MR^2$ the actual curve $J(t)$ initially falls below $f(t)$ until it crosses the parabola somoewhere before its minimum. At the crossing $J(t_c)=f(t_c)$ and $\dot{J}(t_c)\geqslant\dot{f}(t_c)$. From (10) we find that the actual curve lies above the parabola $f(t)$ after passing the crossing point, i.e., $J(t)\geqslant f(t)$ for $t\geqslant t_c$. Thus for $E\geqslant E_0$, $J(t)$ will irreversibly have exceeded the value $3MR^2$, i.e., charges have left the volume of radius R, at a time prior to

$$t_d=2R\sqrt{3M/E_0}, \tag{12}$$

which we take as a measure of the escape or decay time.

In the case $E<E_0$ two circumstances have to be observed. The first one is that at $0<E<E_0$ the negative charge can penetrate the plane S, in other words, a "window" opens in the "wall" S. One can easily calculate (see Fig. 1) that, if the distance between the positive charges is less than d, the "window" will not appear at $E_0<Q(Q-4)/d$. The second difficulty is the fact that the positive force (9) can take any small value and we have to look carefully at the center of mass to predict the decay time and estimate the threshold E_0.

To this end we carry out simple estimates, our aim being to prove only the existence of a time when the system escapes from the volume, independent of initial conditions inside it. Now we assume E_0 to be small enough to have the "window" closed and embed our initial sphere into two other concentric spheres with radii $R<\alpha R<\beta R$. If charge 2 stays inside the second sphere, how far from it can the positive charge 3 be to keep the center of mass at rest (as the center of all three spheres), i.e., where it was initially? If charge 3 moves far away, then, since the negative charge 1 is always closer to 3, the center of mass is pulled out of place unless counterbalanced by the motion of the positive charge 2 in the opposite direction. If charge 2 is placed on the second sphere, the furthest possible position away from it for charge 3 would obviously be obtained in the collinear case with charge 3 (on S) and the center of mass positioned between them. If we denote the position of 3 as βR the CM condition gives

$$-m_2\alpha+m_1\frac{\beta-\alpha}{2}+\beta m_3=0. \tag{13}$$

This gives a linear relation between β and α and it is quite clear that, taking the radius of the third sphere as

$$\beta(\alpha)=\frac{M}{m}\alpha>\frac{2m_2+m_1}{2m_3+m_1}\alpha, \tag{14}$$

i.e., larger than the number derived from Eq. (13), means that if charge 3 is outside the sphere $\beta(\alpha)R$, charge 2 is outside the sphere αR. Below we use the separation measure $D(\alpha)R$, $D(\alpha)=\alpha+\beta(\alpha)=(1+M/m)\alpha$ giving the maximum separation of points on the two reference circles.

Now we are ready to estimate the time of escape. There is always a positive projection of the force (9) acting on charge 2, which makes the radial component of the speed, which we denote by v_2, increase monotonically. We consider charge 3 to be all the time inside the sphere $\beta(1)R$, otherwise due to the center of mass condition charge 2 would leave the initial volume even faster and the monotonic increase in speed would not let it return. In this case, according to Eq. (9) and since $r_{23}\leqslant D(1)R$ we have the inequality

$$\ddot{r}_2\geqslant\frac{Q(Q-4)}{Mr_{23}^2}>\frac{Q(Q-4)}{M[D(1)R]^2}. \tag{15}$$

Now performing similar considerations as for $\ddot{J}$, we just have to modify some constants in Eq. (12) to get the result

$$t_d^{(2)}(R)=\sqrt{\frac{2M}{Q(Q-4)}}\left(\frac{M}{m}+1\right)R^{3/2}. \tag{16}$$

We are left only with setting the threshold E_0 in a way that does not let the "window" open during the motion. Let us put $\alpha=2$ and make the threshold so low that, while charge 2 stays inside the second sphere and charge 3 somewhere inside the third sphere with the radius $\beta(2)R$, i.e., $r_{23}\leqslant D(2)R$, the window remains closed, i.e., we impose the condition

$$E_0\leqslant\frac{Q(Q-4)}{2\left(\frac{M}{m}+1\right)R}\leqslant\frac{Q(Q-4)}{r_{23}}. \tag{17}$$

Now in time $t_d^{(2)}(2R)$ charge 2 would leave the second sphere and since it passed the distance R without losing speed its minimal kinetic energy on the way out from the second sphere has to be

$$T_{min}=\frac{m_2}{2}\left(\frac{R}{t_d^{(2)}(2R)}\right)^2\geqslant\frac{mQ(Q-4)}{32RM(M/m+1)^2}. \tag{18}$$

One can easily check that $E_0=T_{min}$ satisfies (17) and, since the kinetic energy of charge 2 is always larger than T_{min} after it leaves the second sphere, the window remains closed. Now substituting this threshold into Eq. (12) we obtain

$$t_d^{(1)}=\frac{8\sqrt{2}R^{3/2}M}{\sqrt{mQ(Q-4)}}. \tag{19}$$

Thus our estimate for the time of decay is $t_d=\max[t_d^{(1)},t_d^{(2)}]$.

Finally we consider the general case of different charges $(-1,Q_2,Q_3)$. Since all computations are analogous to those above we do not provide complete proofs in what follows. We have to impose the existence of a plane that would prevent the negative charge from coupling together the two positive ones. This is clearly equivalent to the existence of a point between the positive charges where the placement of a negative charge would make the total potential positive. Denoting by x and y the corresponding distances from charges 2 and 3 to the negative charge 1, we demand

$$-\frac{Q_2}{x}-\frac{Q_3}{y}+\frac{Q_2Q_3}{x+y}>0. \tag{20}$$

If we multiply the inequality by $(x+y)$, we obtain

$$(Q_2Q_3-Q_2-Q_3)>Q_2\frac{y}{x}+Q_3\frac{x}{y}. \tag{21}$$

This may be rewritten as

$$(Q_2Q_3-Q_2-Q_3)>\sqrt{Q_2Q_3}\left(s+\frac{1}{s}\right), \tag{22}$$

where $s=\sqrt{Q_2}y/\sqrt{Q_3}x$. The term in square brackets in (22) attains its minimum at $s=1$ and we are left with

$$\sqrt{Q_2Q_3}>\sqrt{Q_2}+\sqrt{Q_3}. \tag{23}$$

We position an instant plane S perpendicular to the line connecting the positive charges at $\sqrt{Q_3}x=\sqrt{Q_2}y$. One can check that on any positive charge there would act a positive radial projection of the force from two other charges remaining behind the plane. Hence the system of charges decays in a finite time when (23) is fullfilled.

In quantum mechanics (QM) the question of stability of three charges has been a subject of extensive studies for a long time. For the system to be stable against decay one needs to have the energy of the system lower than the ground state in any pair channel, which is usually referred to as the threshold in the system. For a wide coverage of results in this area see, for example, the review article [4]. It is known that a system of charges $(-1,Q,Q)$ is always stable at $Q<1$. For $Q=1$ there is a region of mass ratios where the system is stable, and finally at approximately $Q>1.24$ the stability is lost for all masses [5]. Up to now there has been no apparent theoretical attempt to estimate this number [6].

If we take two interacting particles, it is known that a bound state can appear only if the particles can be confined on some energy surface. In the case of three Coulomb charges it is impossible to put them on a compact energy surface due to the singularity of the Coulomb interaction. Thus the stability can be assessed only through the decay time of the system. We do not provide a formal proof but it is intuitively obvious that, if the classical system is decaying in a finite time, the quantum analog system cannot possess bound states. To see this we may expand the wave function into a combination of narrow wave packets, and the timeevolution of this function is largely determined by the Ehrenfest theorem, with all wave packets moving along classical trajectories. Thus the probability density cannot be stationary and the quantum system also decays. Our classical result indicates the quantum instability for all masses at $Q>4$ and it agrees with the results of QM computations predicting instability at $Q>1.24$ regardless of the mass ratio.

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